1. *Let $m \geq 1$ and K a compact subset of $\mathbb{R}^m \times \{0\}$. Then show there is a function $f : \mathbb{R}^{m+1} \to \mathbb{R}$ such that*

   (a) f is continuous in each of the $m+1$ variables separately, and

   (b) f is continuous at $z \in \mathbb{R}^{m+1}$ iff $z \notin K$.

Solution: We denote by $d(x, M)$ the distance of a point $x \in X$ from M whenever M is a closed subset of a metric space (X, d). Recall that this is defined as

$$d(x, M) = \inf_{y \in M} d(x, y).$$

Since K is a compact subset of $\mathbb{R}^{m+1} \times 0$, it must be of the form $K = K' \times \{0\}$ for some compact set $K \subseteq \mathbb{R}^m$. Let $z \in \mathbb{R}^{m+1}, z = (x, y)$, with $x \in \mathbb{R}^m, y \in \mathbb{R}$. We define the function $f : \mathbb{R}^{m+1} \to \mathbb{R}$ as follows:

$$f(z) = f((x,y)) = \begin{cases} \frac{d(x,K')y}{d(z,K)^2}, \ z \notin K \\ 0, \ z \in K. \end{cases}$$

It is clear that f defined this way is continuous outside K. On K, we prove that it is continuous in each of the (m+1) variables separately. Consider the variable x_i for $i \neq m+1$, and a point $z = (x_1^0, \cdots, x_i^0, \cdots, x_m^0, 0) \in K$. Then as $x_i \to x_i^0$ from outside K,

$$\begin{aligned} \lim_{x_i \to x_i^0} f(x_1^0, \cdots, x_i, \cdots, x_m^0, 0) &= \frac{d((x_1^0, \cdots, x_i, \cdots, x_m^0), K)(0)}{d((x_1^0, \cdots, x_i, \cdots, x_m^0, 0), K)^2} \\ &= 0. \end{aligned}$$

Next, we show continuity in the variable x_{m+1}. Let $z = (x_1^0, \cdots, x_m^0, 0) \in K$. Then as $x_{m+1} \to 0$ from outside K,

$$\begin{aligned} \lim_{x_{m+1} \to 0} f(x_1^0, \cdots, x_m^0, 0) &= \frac{d((x_1^0, \cdots, x_m^0), K')(x_{m+1})}{d((x_1^0, \cdots, x_m^0, x_{m+1}), K)^2} \\ &= \frac{0 x_{m+1}}{d((x_1^0, \cdots, x_m^0, x_{m+1}), K)^2}. \\ &= 0. \end{aligned}$$

Hence f is continuous in each of the $m+1$ variables. Finally, we show that f is discontinuous at $z \in K$. It can be seen that for $z \notin K$,

$$\begin{aligned} f(z) = f((x,y)) &= \frac{d(x, K')y}{d(z, K)^2} \\ &= \frac{d(x, K')y}{d(x, K')^2 + y^2} \end{aligned}$$

Choose $y = md(x, K')$. Then as $x \to x_0 \in K'$, $y \to 0$. Then $f(z) = \frac{m}{1+m^2}$, hence proving the discontinuity on K. □

2. *(a) State and prove the chain rule for differentiable functions on domains in Euclidean space.*

   (b) Let $\lambda > 0$, and define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x) = \|x\|^\lambda$. For what values of λ is f continuously differentiable? Justify your answer.

Solution:

(a) See Theorem 9.15 in [1].

(b) We claim that the function $f(x) = \|x\|^\lambda$ is continuously differentiable iff $\lambda > 1$. The only possible point where the function may not be differentiable is at $x = 0$. For $\lambda > 1$,

$$\begin{aligned}\lim_{x \to 0} \frac{|\|x\|^\lambda - 0 \cdot x|}{\|x\|} &= \lim_{x \to 0} \frac{\|x\|^\lambda}{\|x\|} \\ &= \lim_{x \to 0} \|x\|^{\lambda - 1} \\ &= 0.\end{aligned}$$

Hence for $\lambda > 1$, f is differentiable with derivative 0 at the origin. For $\lambda \leq 1$, consider the function $g : \text{Range}\ (f) \to \mathbb{R}$ given by $g(\|x\|^\lambda) = |x_1|^\lambda$, where $x = (x_1, \cdots, x_m)$. As $g \circ f$ is not continuously differentiable and g is continuously differentiable, by the chain rule, f is not continuously differentiable.

3. *(a) Let $x, y, z \in \mathbb{R}^m$ and $a < b < c$ be real numbers. Then show that $\frac{z-x}{c-a}$ belongs to the closed line segment joining $\frac{y-x}{b-a}$ and $\frac{z-y}{c-b}$.*

   (b) Using part (a), show that if $a, b \in \mathbb{R}^m$, f is a differentiable function on a neighbourhood of $[a, b]$ into $\mathbb{R}^n$, then there is an $x \in [a, b]$ such that $\|f(b) - f(a)\| \leq \|b - a\| \|f'(x)\|$.

   (c) Give an example to show that equality may not be possible in part (b).

Solution:

(a) It suffices to find $t \in [0, 1]$ such that

$$(1 - t)\frac{y - x}{b - a} + t\frac{z - y}{c - b} = \frac{z - x}{c - a}. \tag{1}$$

Comparing coefficients of x, y, z in (1) yields $t = \frac{c-b}{c-a}$.

(b) Consider the map ϕ on $[0, 1] \subseteq \mathbb{R}$ given by

$$\phi(t) = (f(b) - f(a)) \cdot (f((1 - t)a + tb)).$$

Then $\phi : \mathbb{R} \to \mathbb{R}$ is differentiable on $(0, 1)$ and continuous on $[0, 1]$, hence we can apply the mean value theorem to obtain $t_0 \in [0, 1]$ such that

$$\phi(1) - \phi(0) = (f(b) - f(a)) \cdot f'((1 - t_0)a + t_0 b)(b - a).$$

On the other hand,

$$\phi(1) - \phi(0) = \|f(b) - f(a)\|^2.$$

Hence, by applying the Cauchy Schwarz inequality in $\mathbb{R}^n$, we get

$$\|f(b) - f(a)\|^2 \leq \|b - a\| \|f(b) - f(a)\| \|f'((1 - t_0)a + t_0 b)\|.$$

As $(1 - t_0)a + t_0 b \in [a, b]$, we are done.

(c) Let $f : [0, 2\pi] \to \mathbb{R}^2$ be defined by $f(x) = (\cos x, \sin x)$. Then $f'(x) = \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix}$. Hence $\|f'(x)\|^2 = 1 \,\forall x \in (0, 2\pi)$. On the other hand, $\|f(2\pi) - f(0)\|^2 = 0$.

4. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable. Suppose $f'(x)$ is invertible for all $x \in \mathbb{R}^m$. Then show that f is an open map, that is, $f(U)$ is open for all open subsets U of $\mathbb{R}^m$.*

Solution: Let $U \in \mathbb{R}^m$ be an open set. By the inverse function theorem, for each $x \in U$, there exists open sets $U_x \subset U$ and V_x such that $f(U_x) = V_x$. Since $f(U) = f(\cup_{x \in U} U_x) = \cup_{x \in X} V_x$, $f(U)$ is open.

5. *Let $GL(n, \mathbb{R})$ denote the group of all invertible linear operators on $\mathbb{R}^n$ to $\mathbb{R}^n$. Show that the function $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ defined by $f(A) = A^{-1}$ is continuous.*

Solution: Note that for $A, B \in GL(n, \mathbb{R})$,

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.$$

Hence

$$\|A^{-1} - B^{-1}\| \leq \|A^{-1}\|\|B^{-1}\|\|B - A\|.$$

Thus if $A_k, A \in GL(n, \mathbb{R})$ and $A_k \to A$, then

$$\|A^{-1} - A_k^{-1}\| \leq \|A^{-1}\| \sup_k \|A_k^{-1}\|\|A - A_k\|.$$

Thus f is continuous.

6. *Give an example of a function $f : \mathbb{R}^n \to \mathbb{R}$ such that all the partial derivatives of f exist everywhere but f is not differentiable. Justify your answer.*

Solution: Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as follows:

$$f(x_1, \cdots, x_n) = \begin{cases} \frac{x_1 \cdots x_n}{x_1^n + \cdots x_n^n}, \ (x_1, \cdots, x_n) \neq 0 \\ 0, \ (x_1, \cdots, x_n) = 0. \end{cases}$$

Then the partial derivative of f with respect to each x_i exists but f is not continuous at the origin and hence not differentiable.

It is easy to see that each partial derivative at the origin is given by:

$$\frac{\partial f}{\partial x_i}(\mathbf{0}) = \lim_{\mathbf{h} \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{0}, \cdots, \mathbf{h}, \mathbf{0}, \cdots, \mathbf{0}) - \mathbf{f}(\mathbf{0})}{\mathbf{h}} = \mathbf{0}.$$

On the other hand, choosing the path $x_i = mx_1$ for $i \neq 1$ and $x_1 \to 0$ gives

$$\lim_{(x_1, \cdots, x_n) \to 0} f(x_1, \cdots, x_n) = \frac{m^{n-1}}{1 + (n-1)m^n}.$$

Hence the function is not even continuous at the origin.

References

[1] Rudin, Walter. "Principles of Mathematical Analysis (International Series in Pure and Applied Mathematics)." (1976).